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Abundance of invariant equations and properties for wave propagation in a space of arbitrary dimensions

J Heading

Department of Applied Mathematics, The University College of Wales, Penglais, Aberystwyth, Dyfed SY23 3BZ, UK

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Abstract. This paper continues earlier work on the generalisation of the differential equations governing the propagation of electromagnetic waves in an inhomogeneous ionised plasma in a space of arbitrary (odd) dimensions. Using a convenient definition of curl in such a space, together with appropriate definitions of electric and magnetic field components, Maxwell's first-order equations are defined together with the corresponding second-order wave equation involving the generalised curl curl operator. Various forms of these equations are discussed when a suitable plane of incidence is defined. The property of invariance is introduced, under which various forms of the equations and deductions therefrom are independent of the dimensions of the space in which propagation is defined to occur; both isotropic and anisotropic media are discussed.

1. The nature of the investigation

In spatial terms, physical problems are expressed in the space of three dimensions R^3. This is, of course, completely satisfactory when the object of the mathematical investigations is to explain observed phenomena or to predict the results of untried experiments. But the restriction of the mathematical exercise to the space R^3 means that the actual form of the associated equations as a function of dimensionality is fixed. For comparison purposes, there is no variation in the structure of the equations nor in the results deduced from them. An extension of the equations to a space of an arbitrary number of dimensions permits the examination of an overall structure; the form of any particular equation can then be seen to be a member of the structure. The same may be said about deductions: the actual physical results form an elementary base of a structure that often is of grand proportions.

Particular interest is evoked when there is a simple pattern amongst the hierarchy of equations and results that arise. More particularly, a fundamental property of invariance may arise with respect to the dimensionality of the space in which propagation is defined. We do not use the term 'invariance' in the context of tensor analysis relative to a space of given dimensionality; rather it is used of the form of equations and results that have the same properties in spaces of *all* dimensionalities (sometimes restricted to an odd number). The starting point is, of course, any appropriate fundamental physical equation in R^3. Generalisation may be possible in several directions; either all, or only some, of these suggested generalisations may throw up the property of invariance as the investigation proceeds.

These remarks follow from the author's previous investigations into wave propagation in inhomogeneous media, where understanding of the basic physical and mathematical processes has been enhanced by many excursions into spaces of higher dimensions, or by the use of differential equations and/or matrices of general order. Usually, an examination of the physical case itself gives no clue as to the properties of the generalisation.

The fundamental equations governing electromagnetic wave propagation in isotropic and anisotropic stratified media may be found in texts by Bremmer (1949), Ratcliffe (1959), Ginzburg (1961) and Budden (1961). Many generalisations have already been considered by the present author over the past decade. Generalised reciprocity relations, and conditions for their existence, are considered in Heading (1973a, 1975a). General energy flux invariants are dealt with in Heading (1975b), while the actual forms of the equations under both isotropic and anisotropic conditions are derived in Heading (1976a). The present paper extends this latter work by generalising the equations in a different and more appropriate direction. Two-way transmission is considered in Heading (1977, 1978a, 1979), using both exact and approximate methods. A wide set of results is derived when propagation is governed by self-adjoint and Hermitian self-adjoint differential operators of general even order (Heading 1978b), while the separation of the medium approximately into non-overlapping domains is considered in Heading (1980). The expansion of a reflection coefficient in terms of a small parameter is examined in Heading (1981), showing how the corresponding results for a second-order equation are generalised. Finally, generalised zilch flux and zilch density are generalised in two papers (Heading 1973b, 1975d).

In Heading (1976b), the analogy to the triple vector product in a space of arbitrary dimensions is discussed, the result being a vector, although the intermediate stage of the analogy to the vector product is not a vector. This means that, using the vector differential operator ∇ in such a general space, there exists a wave equation involving curl curl (with curl not being defined as a vector). This generalisation to second-order equations is exploited in Heading (1975a, 1976a). The drawback to this, however, is that the operator curl (not being a vector) does not lead to suitable generalisations of Maxwell's first-order equations for the electric and magnetic field components. The present paper is designed to overcome this difficulty.

By means of a suitably defined generalisation of the operator curl, Maxwell's first-order equations can be extended in form to a space of arbitrary (odd) dimensions. In turn, these lead to an appropriate second-order wave equation, the operator involving more terms that those appearing in the author's papers (Heading 1975a, 1976a). When a suitable plane of incidence is defined for incident waves, the set of equations is separated into three distinct types of equations in an isotropic medium, two of which generalise the familiar horizontally and vertically polarised waves in physical space. In fact, invariant forms emerge from the mathematical structure that is built up from the definitions, meaning that many of the results pertaining to physical space can be imported directly into the hierarchy pertaining to generalised space. The generalised complex Poynting vector is also defined, and it is shown that it is constant when the medium contains no energy-loss terms. Anisotropic propagation is discussed in a restricted way, corresponding to the circular polarisation properties of waves propagating in R^3 perpendicular to the planes of stratification when the externally maintained magnetic field is also in this direction.

2. Results to be generalised

When the time factor $\exp(\mathrm{i}\omega t)$ is suppressed, Maxwell's equations in R^3 relating to an isotropic medium take the form

$$\operatorname{curl} \boldsymbol{e} = -\mathrm{i}\omega\mu_0 \boldsymbol{h}, \tag{1}$$

$$\operatorname{curl} \boldsymbol{h} = \mathrm{i}\omega\varepsilon_0 \boldsymbol{e} - \mathrm{i}\omega\varepsilon_0 m \boldsymbol{e}, \tag{2}$$

where m denotes a function of position. When $\boldsymbol{h}$ is eliminated, we obtain the second-order equation

$$\operatorname{curl}\operatorname{curl} \boldsymbol{e} = k^2(1-m)\boldsymbol{e}, \tag{3}$$

where $k^2 = \varepsilon_0\mu_0\omega^2$, namely,

$$\operatorname{grad}\operatorname{div} \boldsymbol{e} - \boldsymbol{\nabla}^2\boldsymbol{e} = k^2(1-m)\boldsymbol{e}. \tag{4}$$

In Heading (1975a, 1976a), the author has used this equation to define an appropriate generalisation to the second-order wave equation in R^n, namely

$$\boldsymbol{\nabla}\boldsymbol{\nabla}^{\mathrm{T}}\boldsymbol{e} - \boldsymbol{\nabla}^{\mathrm{T}}\boldsymbol{\nabla}\boldsymbol{e} = k^2(1-m)\boldsymbol{e}$$

in matrix notation, T denoting the transpose, and $\boldsymbol{\nabla}$ and $\boldsymbol{e}$ being n-vectors; in the anisotropic case, $1-m$ is more generally replaced by the matrix $\boldsymbol{I}-\boldsymbol{M}$. However, this does not permit an analogue to the magnetic field $\boldsymbol{h}$ to be defined in R^n, since in R^n there is no *first*-order differential tensor operator corresponding to curl in R^3, such that the operator, when operating on a vector, yields a vector. In Heading (1976a) only a pseudo-magnetic field with $n-1$ components could be defined when the medium was stratified with respect to one coordinate. In Heading (1976b), the author considered the analogue to the triple vector product in R^n, the half-way stage (that is, corresponding to the vector product in R^3) being introduced by means of a dual tensor; this idea is extended in the present generalisation.

In this section, we now recall the equations in R^3 that are to be examined in connection with their generalisation to a space of an odd number of dimensions. With coordinates x, y, z, and with m defined to be a function of z alone, equations (1), (2) and (4) admit separated solutions containing the x factor $\exp(-\mathrm{i}kSx)$ and with no y factor. Equation (4) yields

$$-\partial_x\partial_z e_z + \partial_z^2 e_x + k^2(1-m)e_x = 0, \tag{5}$$

$$(\partial_x^2 + \partial_z^2)e_y + k^2(1-m)e_y = 0, \tag{6}$$

$$-\partial_x\partial_z e_x + \partial_x^2 e_z + k^2(1-m)e_z = 0, \tag{7}$$

where ∂/∂_x must be replaced by $-\mathrm{i}kS$. Hence there are two independent modes of isotropic propagation, the e_y field being horizontally polarised and the e_x, e_z field vertically polarised.

Since $h_y \propto \partial_z e_x - \partial_x e_z$, equations (5) and (7) can be combined to yield

$$\mathrm{d}_z^2 h_y + [\mathrm{d}_z m/(1-m)]\,\mathrm{d}_z h_y + k^2(C^2-m)h_y = 0, \tag{8}$$

where $C^2 = 1-S^2$ and $\mathrm{d}_z \equiv \mathrm{d}/\mathrm{d}z$. Again, using e_z from (7), we eliminate e_z from (5), yielding the more complicated equation for e_x:

$$\mathrm{d}_z^2 e_x + \{S^2\mathrm{d}_z m/[(1-m)(C^2-m)]\}\mathrm{d}_z e_x + k^2(C^2-m)e_x = 0. \tag{9}$$

Equations (8) and (9) are the second-order equations for h_y and e_x in the vertically polarised mode. From (6), the equation for e_y in the horizontally polarised mode is

$$\mathrm{d}_z^2 e_y + k^2(C^2 - m)e_y = 0. \tag{10}$$

When $\boldsymbol{e}$ is a vector in R^n, we have seen in Heading (1976a) that the form of equations (8), (9) and (10) remains unchanged, namely the forms are invariants with respect to n.

When $\boldsymbol{M}$ is a matrix in R^3 in the anisotropic case, it effectively is of the form

$$\boldsymbol{M} = \alpha_0 \boldsymbol{I} + \alpha_1 \boldsymbol{N} + \alpha_2 \boldsymbol{N}^2, \tag{11}$$

where

$$\boldsymbol{N} = \begin{pmatrix} 0 & -n & m \\ n & 0 & -l \\ -m & l & 0 \end{pmatrix},$$

and $\boldsymbol{n} = (l, m, n)$ denotes the direction cosines of the constant applied magnetic field. Since $\boldsymbol{N}^3 = -\boldsymbol{N}$, no higher powers of $\boldsymbol{N}$ occur in $\boldsymbol{M}$. This special form (11) for $\boldsymbol{M}$ does not appear to have been recognised by writers on propagation in ionised anisotropic media, but it is essential for our generalisation.

When $S = 0$ (vertical propagation), with $l = m = 0$, $n = 1$ (vertical magnetic field),

$$\boldsymbol{M} = \begin{pmatrix} \alpha_0 - \alpha_2 & -\alpha_1 & 0 \\ \alpha_1 & \alpha_0 - \alpha_2 & 0 \\ 0 & 0 & \alpha_0 \end{pmatrix},$$

so equation (3) yields

$$\left.\begin{aligned} &\mathrm{d}_z^2 e_x + k^2[e_x - (\alpha_0 - \alpha_2)e_x + \alpha_1 e_y] = 0, \\ &\mathrm{d}_z^2 e_y + k^2[e_y - \alpha_1 e_x - (\alpha_0 - \alpha_2)e_y] = 0. \end{aligned}\right\} \tag{12}$$

The combinations $u = e_x + \mathrm{i}e_y$ and $v = e_x - \mathrm{i}e_y$ yield non-simultaneous equations

$$\left.\begin{aligned} &\mathrm{d}_z^2 u + k^2(1 - \alpha_0 + \alpha_2 - \mathrm{i}\alpha_1)u = 0, \\ &\mathrm{d}_z^2 v + k^2(1 - \alpha_0 + \alpha_2 + \mathrm{i}\alpha_1)v = 0. \end{aligned}\right\} \tag{13}$$

With respect to the x and y axes in R^3, the u and v fields are circularly polarised. The u field existing alone demands $v = 0$ and $e_x = \mathrm{i}e_y$, which is the condition for circular polarisation, namely $|e_x| = |e_y|$ and $\arg e_x - \arg e_y = \frac{1}{2}\pi$.

When the equations are generalised to R^n according to the ideas developed in the author's (1976a) paper, it has been proved that the property of circular polarisation of many fields passes over into R^n. When n is even, one field is linearly polarised.

3. An appropriate definition of curl in R^{2n+1}

Let e_i denote a vector in Cartesian space R^{2n+1}. The rotation of orthogonal axes is governed by an orthogonal matrix, so no distinction exists between contravariance and covariance; we shall therefore use suffixes throughout, except in δ symbols. The summation convention is also used throughout the investigation.

In R^{2n+1}, curl $\boldsymbol{e}$ is usually defined to be a skewsymmetric tensor of order two:

$$E_{ij} = \partial_i e_j - \partial_j e_i,$$

possessing $n(2n+1)$ independent components. Its divergence

$$\begin{aligned}\partial_j E_{ij} &= \partial_j \partial_i e_j - \partial_j \partial_j e_i \\ &\equiv \text{grad div } \boldsymbol{e} - \nabla^2 \boldsymbol{e}\end{aligned}$$

is a vector equivalent to the standard identity for curl curl $\boldsymbol{e}$ when $n = 1$. As we have observed, this is used in Heading (1975a, 1976a), but the corresponding magnetic field is artificial as already explained.

A complete generalisation requires the introduction of a first-order differential tensor of order S operating on a tensor of order T, yielding by contraction a tensor of order $S - T$. So that this operator of order S can similarly operate again on this contracted tensor, we require $S - T = T$, or $T = \frac{1}{2}S$, so S must be an even integer. In R^3, we have $S = 2$, $T = 1$ (for a vector), so generally we may deliberately choose $S = 2n$, $T = n$ for the space R^{2n+1}, which will enable the skewsymmetry of the matrix $\boldsymbol{N}$ to be retained in the generalisation.

In R^{2n+1}, we choose the analogue to the electric field to be a completely skewsymmetric Cartesian tensor $e_{ij\ldots l}$ of order n (that is, skewsymmetric with respect to all pairs of suffixes), possessing $(2n+1)!/n!(n+1)!$ independent components. The analogue to the magnetic field is similarly chosen to be a completely skewsymmetric tensor $h_{ij\ldots l}$ or order n.

The x_{2n+1} axis will be regarded as 'vertical' for designation purposes only, and the plane of incidence will be defined by the x_1 and x_{2n+1} axes. Relative to this plane and vertical axis, some of the components of $e_{ij\ldots l}$ and of $h_{ij\ldots l}$ are classified as possessing the property of horizontal polarisation, others as possessing the property of vertical polarisation, while the remainder possess the property of 'incident-plane' polarisation; this latter property cannot exist in R^3 when $n = 1$. Horizontally polarised components $e_{ij\ldots l}$ are those for which the n integers $ij\ldots l$ do not include 1 and $2n + 1$. The number of such independent components is the number of ways in which $2n - 1$ numbers may be arranged amongst n positions, namely ${}_{2n-1}C_n$. A vertically polarised electric field will consist of a pair of electric components, namely $e_{1j\ldots l}$ and $e_{2n+1,j\ldots l}$, where $j \ldots l$ is a combination of the integers $2, 3, \ldots, 2n$, not involving 1 and $2n + 1$. The number of such pairs is ${}_{2n-1}C_{n-1}$. The 'incident-plane' polarised components $e_{ij\ldots l}$ are those for which both 1 and $2n + 1$ are included in the suffixes; there are ${}_{2n-1}C_{n-2}$ such components. This classification accounts for all components, since

$${}_{2n-1}C_n + 2\,{}_{2n-1}C_{n-1} + {}_{2n-1}C_{n-2} \equiv {}_{2n+1}C_n.$$

We now use the usual permutation symbol $\varepsilon_{ij\ldots lpab\ldots f}$ with $2n + 1$ suffixes.

The tensor operator curl is defined to be the skewsymmetric tensor operator of order $2n$ (of order one as far as differentiation is concerned):

$$C_{ij\ldots lab\ldots f} \equiv (1/n!)\varepsilon_{ij\ldots lpab\ldots f}\,\partial_p, \tag{14}$$

where $ij\ldots l$ and $ab\ldots f$ denote any sets of n integers chosen from $1, 2, \ldots, 2n + 1$.

This generalises the skewsymmetric tensor of order 2 in R^3:

$$C_{ia}=\varepsilon_{ipa}\partial_p=\begin{pmatrix}0 & -\partial_3 & \partial_2\\ \partial_3 & 0 & -\partial_1\\ -\partial_2 & \partial_1 & 0\end{pmatrix}.$$

$C_{ij\ldots lab\ldots f}$ operates on a skewsymmetric tensor of order n, producing by contraction on the symbols $ab\ldots f$ another skewsymmetric tensor of order n. Thus in R^3 we have

$$C_{ia}e_a=\varepsilon_{ipa}\partial_p e_a=(\operatorname{curl}\boldsymbol{e})_i.$$

4. Maxwell's equations generalised

With the time factor $\exp(\mathrm{i}\omega t)$ suppressed, we now define the isotropic generalisation to consist of the $2(2n+1)!/n!(n+1)!$ partial differential equations

$$C_{ij\ldots lab\ldots f}e_{ab\ldots f}=[(-1)^n\mathrm{i}\omega\mu_0/n!]\delta^{ab\ldots f}_{ij\ldots l}h_{ab\ldots f},\tag{15}$$

$$C_{xy\ldots zij\ldots l}h_{ij\ldots l}=(\mathrm{i}\omega\varepsilon_0/n!)(1-m)\delta^{ij\ldots l}_{xy\ldots z}e_{ij\ldots l},\tag{16}$$

where the δ symbol denotes the generalised Kronecker delta of order $2n$, and the factor $(-1)^n$ is introduced in anticipation of the subsequent results. When $n=1$, the Kronecker delta is merely the unit matrix of order three. Here, m is a function of position, such that $m=0$ defines what is to be understood as free space.

First we carry out the summation implied on the right-hand sides of (15) and (16). In (15), for any particular choice of values $ij\ldots l$, $ab\ldots f$ must be restricted to these values, but the number of combinations will be $n!$, and all allowable products $\delta^{ab\ldots f}_{ij\ldots l}h_{ab\ldots f}$ with these combinations will all possess the same sign. Then

$$C_{ij\ldots lab\ldots f}e_{ab\ldots f}=(-1)^n\mathrm{i}\omega\mu_0 h_{ij\ldots l},\tag{17}$$

$$C_{xy\ldots zij\ldots l}h_{ij\ldots l}=\mathrm{i}\omega\varepsilon_0(1-m)e_{xy\ldots z}.\tag{18}$$

To obtain the second-order equation satisfied by the field $e_{ab\ldots f}$, we eliminate $h_{ij\ldots l}$ by repeating the operation $C_{xy\ldots zij\ldots l}$, giving

$$C_{xy\ldots zij\ldots l}C_{ij\ldots lab\ldots f}e_{ab\ldots f}=(-1)^{n+1}k^2(1-m)e_{xy\ldots z}.$$

In terms of the differential operators ∂_s, this equation is

$$\frac{1}{(n!)^2}\varepsilon_{xy\ldots zsij\ldots l}\varepsilon_{ij\ldots ltab\ldots f}\partial_s\partial_t e_{ab\ldots f}=(-1)^{n+1}k^2(1-m)e_{xy\ldots z}.\tag{19}$$

We replace $\varepsilon_{ij\ldots ltab\ldots f}$ by $(-1)^n\varepsilon_{ab\ldots ftij\ldots l}$, and use the identity

$$\varepsilon_{xy\ldots zsij\ldots l}\varepsilon_{ab\ldots ftij\ldots l}\equiv n!\,\delta^{xy\ldots zs}_{ab\ldots ft};$$

see Korn and Korn (1961). Then equation (19) becomes

$$\frac{1}{n!}\delta^{xy\ldots zs}_{ab\ldots ft}\partial_s\partial_t e_{ab\ldots f}=-k^2(1-m)e_{xy\ldots z}.\tag{20}$$

5. Reduction of the second-order wave equation

In equation (20), the suffix t must equal one of the superscripts $xy\ldots zs$. If $t=s$, where $s=1,2,\ldots,2n+1$ but excluding $xy\ldots z$, we have

$$
\begin{aligned}
&\frac{1}{n!}\delta^{xy\ldots zs}_{ab\ldots ft}\partial_s^2 e_{ab\ldots f}\\
&\qquad=\sum_{s=1}^{2n+1}\partial_s^2 e_{xy\ldots z}\qquad\text{(excluding } s=xy\ldots z)\\
&\qquad=\nabla^2 e_{xy\ldots z}-(\partial_x^2+\partial_y^2+\ldots+\partial_z^2)e_{xy\ldots z}.
\end{aligned}\tag{21}
$$

In the sum over t in (20), this suffix can now only equal $xy\ldots z$. If, for example, $t=x$, with $s=1,2,\ldots,2n+1$ but excluding $xy\ldots z$, the contribution to the left-hand side of (20) is

$$
\begin{aligned}
&\delta^{xy\ldots zs}_{ab\ldots fx}\partial_s\partial_x e_{ab\ldots f}\qquad\text{(not summed over } ab\ldots f,\, x)\\
&\qquad=(-1)^n\delta^{xy\ldots zs}_{xab\ldots f}\partial_s\partial_x e_{ab\ldots f}\\
&\qquad=(-1)^n\delta^{xy\ldots zs}_{xy\ldots zs}\partial_s\partial_x e_{y\ldots zs}\\
&\qquad=(-1)^n\partial_s\partial_x e_{y\ldots zs}\qquad(s\neq xy\ldots z)\\
&\qquad=(-1)^n\partial_x\left[\sum_{\text{all } s}\partial_s e_{y\ldots zs}-\partial_x e_{y\ldots zx}-\partial_y e_{y\ldots zy}-\ldots\right]\\
&\qquad=(-1)^n\partial_x\partial_s e_{y\ldots zs}-(-1)^n\partial_x^2 e_{y\ldots zx}\\
&\qquad=(-1)^n(-1)^{n-1}\partial_x\partial_s e_{sy\ldots z}-(-1)^n(-1)^{n-1}\partial_x^2 e_{xy\ldots z}.
\end{aligned}\tag{22}
$$

Similarly, for $t=y$ the contribution to (20) is

$$
(-1)^{n-1}(-1)^{n-2}\partial_y\partial_s e_{xs\ldots z}-(-1)^{n-1}(-1)^{n-2}\partial_y^2 e_{xy\ldots z},\tag{23}
$$

and so on. Hence the complete contribution from (21), (22) and (23) is

$$
\begin{aligned}
&\nabla^2 e_{xy\ldots z}-(\partial_x^2+\partial_y^2+\ldots+\partial_z^2)e_{xy\ldots z}-\partial_x\partial_s e_{sy\ldots z}+\partial_x^2 e_{xy\ldots z}-\partial_y\partial_s e_{xs\ldots z}+\partial_y^2 e_{xy\ldots z}-\ldots\\
&\qquad=\nabla^2 e_{xy\ldots z}-\partial_x\partial_s e_{sy\ldots z}-\partial_y\partial_s e_{xs\ldots z}-\ldots-\partial_z\partial_s e_{xy\ldots s},
\end{aligned}
$$

this being the generalisation of the expansion of curl curl e in R^3, an expansion that we have not noticed before in the literature.

Finally, equation (20) becomes

$$
\partial_x\partial_s e_{sy\ldots z}+\partial_y\partial_s e_{xs\ldots z}+\ldots+\partial_z\partial_s e_{xy\ldots s}-\nabla^2 e_{xy\ldots z}=k^2(1-m)e_{xy\ldots z},\tag{24}
$$

being the appropriate generalisation of equation (3) when $n=1$. The tensor h can then be determined from equation (10) when the components of e are known.

6. Vertical incidence

We now regard the medium as stratified, in the sense that m is a function of x_{2n+1} only, a direction regarded as vertical. Particular solutions of (24) exist such that all components are functions of x_{2n+1} only. Hence only $\partial/\partial x_{2n+1}$ yields a non-zero derivative.

If in (24) the suffixes $xy\ldots z$ *do not contain* $2n+1$, then evidently

$$d_{2n+1}^2 e_{xy\ldots z} + k^2(1-m)e_{xy\ldots z} = 0,$$

identical in form for all values of n; it is identical to the vertical incidence equations for e_1 and e_2 when $n=1$. But if the suffixes $xy\ldots z$ *do contain* $2n+1$, then

$$0 = k^2(1-m)e_{xy\ldots z},$$

so generally $e_{xy\ldots z}$ vanishes under these circumstances, identical to the property when $n=1$ that the electric field has no component perpendicular to the stratifications. These are therefore invariant equations and properties.

From (10) under these circumstances, with a general value of μ, we have

$$h_{ij\ldots l} \propto C_{ij\ldots lab\ldots f} e_{ab\ldots f}$$
$$= \varepsilon_{ij\ldots lsab\ldots f} \partial_s e_{ab\ldots f},$$

so

$$h_{ij\ldots l} \propto \mu^{-1} d_{2n+1} e_{ab\ldots f}$$

where $ab\ldots f \neq ij\ldots l$, $2n+1$. We define the suffixes of h to be complementary to those of e with respect to $2n+1$. If the boundary conditions are such that $e_{ab\ldots f}$ and $h_{ij\ldots l}$ (with complementary suffixes with respect to $2n+1$) are continuous over a boundary separating two distinct homogeneous media, then it is obvious that the same Fresnel reflection and transmission coefficients emerge for vertical incidence in R^{2n+1} as in R^3. These coefficients are therefore invariants, independent of n.

This field satisfies the condition

$$e_{ij\ldots l} h_{ij\ldots l} = 0$$

for the special case of complementary suffixes, since all the h-components vanish. A rotation of axes about x_{2n+1} yields

$$e'_{ij\ldots l} h'_{ij\ldots l} = 0$$

where the components do not vanish. This is a generalisation of the fact that $\boldsymbol{e}$ and $\boldsymbol{h}$ are perpendicular in R^3 (i) generally in a homogeneous isotropic medium, and (ii) when propagation is perpendicular to the stratifications in an inhomogeneous isotropic medium.

7. Oblique incidence

The plane of incidence is defined by the x_1 and x_{2n+1} axes. Separated solutions of equation (24) are sought, containing functions of x_1 and x_{2n+1} only. The x_1 factor will be of the form $\exp(-ikSx_1)$, with $\partial/\partial x_1 = -ikS$.

Case (i). When the suffixes $xy\ldots z$ *do not contain* 1 and $2n+1$, all the derivatives $\partial_x, \partial_y, \ldots, \partial_z$ vanish, yielding

$$(\partial_1^2 + \partial_{2n+1}^2)e_{xy\ldots z} + k^2(1-m)e_{xy\ldots z} = 0,$$

or

$$d_{2n+1}^2 e_{xy\ldots z} + k^2(C^2 - m)e_{xy\ldots z} = 0, \tag{25}$$

being the equation for all horizontally polarised components (as defined in paragraph 3). They all propagate independently; there are ${}_{2n-1}C_n$ such components.

Case (ii). When $xy\ldots z$ *contain* both 1 and $2n+1$, the left-hand side of (24) vanishes, yielding $e_{1\ldots 2n+1}=0$. The 'incident-plane' polarised components therefore vanish; there are ${}_{2n-1}C_{n-2}$ such components.

Case (iii). In equation (24), let $x=1$, but with $2n+1$ excluded from the remaining suffixes. Then, since $\partial_y=\ldots\partial_z=0$,

$$\partial_1(\partial_1 e_{1y\ldots z}+\partial_{2n+1}e_{2n+1,y\ldots z})-(\partial_1^2+\partial_{2n+1}^2)e_{1y\ldots z}=k^2(1-m)e_{1y\ldots z},$$

or

$$-\partial_1\partial_{2n+1}e_{2n+1,y\ldots z}+\partial_{2n+1}^2 e_{1y\ldots z}+k^2(1-m)e_{1y\ldots z}=0. \tag{26}$$

Similarly, when $x=2n+1$ but with 1 excluded from the remaining suffixes, (24) yields

$$\partial_{2n+1}(\partial_1 e_{1y\ldots z}+\partial_{2n+1}e_{2n+1,y\ldots z})-(\partial_1^2+\partial_{2n+1}^2)e_{2n+1,y\ldots z}$$
$$=k^2(1-m)e_{2n+1,y\ldots z},$$

or

$$-\partial_{2n+1}\partial_1 e_{1y\ldots z}+\partial_1^2 e_{2n+1,y\ldots z}+k^2(1-m)e_{2n+1,y\ldots z}=0. \tag{27}$$

These are the equations for the components of a vertically polarised field, of which there are ${}_{2n-1}C_{n-1}$ pairs.

Equations (26) and (27) are identical with equations (5) and (7) valid when $n=1$, while equation (25) is identical with (6). Hence the forms of these wave equations are independent of n. This means that equation (9), valid when $n=1$ for e_x, is also valid for $e_{1y\ldots z}$ in R^{2n+1}.

The magnetic component corresponding to the pair $e_{1y\ldots z}$ and $e_{2n+1,y\ldots z}$ is easily found. From (10), we have

$$\begin{aligned} h_{ij\ldots l} &\propto C_{ij\ldots lab\ldots f}e_{ab\ldots f}\\ &\propto \varepsilon_{ij\ldots lsab\ldots f}\partial_s e_{ab\ldots f}\\ &=\varepsilon_{ij\ldots l1ab\ldots f}\partial_1 e_{ab\ldots f}+\varepsilon_{ij\ldots l,2n+1,ab\ldots f}\partial_{2n+1}e_{ab\ldots f}.\end{aligned}$$

Let $ij\ldots l$ denote the suffixes arising from $1,2,\ldots,2n+1$ when $1,y,\ldots,z,2n+1$ are deleted, implying that $ab\ldots f$ must equal $1y\ldots z$ and $2n+1,y\ldots z$; hence

$$\begin{aligned} h_{ij\ldots l} &\propto \varepsilon_{ij\ldots l1,2n+1,y\ldots z}\partial_1 e_{2n+1,y\ldots z}+\varepsilon_{ij\ldots l,2n+1,1y\ldots z}\partial_{2n+1}e_{1y\ldots z}\\ &\propto -\partial_1 e_{2n+1,y\ldots z}+\partial_{2n+1}e_{1y\ldots z},\end{aligned}$$

corresponding to h_y in R^3. Hence equation (8) is also quite generally the equation for $h_{ij\ldots l}$.

This investigation therefore shows that equations (8), (9) and (10), valid in R^3 (namely, for the horizontal magnetic and electric components in the vertically polarised mode, and for the horizontal component of the electric field in the horizontally polarised mode), and not special forms just for $n=1$, but they are invariant for all n and generally applicable. Hence any deductions drawn from these equations, notably the reflection and transmission coefficients relating to a given model $m(x_{2n+1})$, are also independent of n. The theory of the critical and Brewster angles is also invariant.

8. The generalised complex Poynting vector

We consider the vector

$$N_s = (n!)^{-2}\varepsilon_{sab\ldots fij\ldots l}e_{ab\ldots f}h_{ij\ldots l}$$

in its complex form. We combine (17) multiplied by $h^*_{ij\ldots l}$ and (18) multiplied by $e^*_{xy\ldots z}$, yielding

$$-(-1)^n h^*_{ij\ldots l}\varepsilon_{ij\ldots lsab\ldots f}\partial_s e_{ab\ldots f} + \mathrm{i}\omega\mu_0 n!\, h_{ij\ldots l}h^*_{ij\ldots l}$$
$$- e^*_{ab\ldots f}\varepsilon_{ab\ldots fsij\ldots l}\partial_s h_{ij\ldots l} + \mathrm{i}\omega\varepsilon_0 n!(1-m)e_{ab\ldots f}e^*_{ab\ldots f} = 0,$$

with a convenient change of suffixes in the last two terms. The first ε symbol is changed to $(-1)^n\varepsilon_{ab\ldots fsij\ldots l}$. The $n!$ cancels if we now use only combinations of n suffixes instead of their permutations. To indicate this, we use the symbol $(ij\ldots l)$ to denote one particular combination instead of the $n!$ permutations. Then

$$-\varepsilon_{(ab\ldots f)s(ij\ldots l)}(h^*_{(ij\ldots l)}\partial_s e_{(ab\ldots f)} + e^*_{(ab\ldots f)}\partial_s h_{(ij\ldots l)})$$
$$+\mathrm{i}\omega[\mu_0 h_{(ij\ldots l)}h^*_{(ij\ldots l)} + \varepsilon_0(1-m)e_{(ij\ldots l)}e^*_{(ij\ldots l)}] = 0.$$

We now take the real part of this equation (by forming the sum of a complex number and its conjugate),

$$-\tfrac{1}{2}\varepsilon_{(ab\ldots f)s(ij\ldots l)}(h^*_{(ij\ldots l)}\partial_s e_{(ab\ldots f)} + h_{(ij\ldots l)}\partial_s e^*_{(ab\ldots f)}$$
$$+ e^*_{(ab\ldots f)}\partial_s h_{(ij\ldots l)} + e_{(ab\ldots f)}\partial_s h^*_{(ij\ldots l)})$$
$$+\omega\varepsilon_0(\mathrm{Im}\, m)e_{(ij\ldots l)}e^*_{(ij\ldots l)} = 0,$$

leading to

$$-\partial_s\varepsilon_{(ab\ldots f)s(ij\ldots l)}\,\mathrm{Re}(e_{(ab\ldots f)}h^*_{(ij\ldots l)}) + \omega\varepsilon_0(\mathrm{Im}\, m)e_{(ij\ldots l)}e^*_{(ij\ldots l)} = 0.$$

We interpret this is an energy balance equation; see Heading (1975c, chap 3). In particular, if $\mathrm{Im}\, m = 0$,

$$-\partial_s\varepsilon_{(ab\ldots f)s(ij\ldots l)}\,\mathrm{Re}(e_{(ab\ldots f)}h^*_{(ij\ldots l)}) = 0.$$

Using the incident plane as defined by the x_1 and x_{2n+1} axes, we note that the operator ∂_1 yields no contribution, since the factor $\exp(-\mathrm{i}kSx_1)$ disappears in $e_{(ab\ldots f)}h^*_{(ij\ldots l)}$, implying that

$$-\varepsilon_{(ab\ldots f),2n+1,(ij\ldots l)}\,\mathrm{Re}\; e_{(ab\ldots f)}h^*_{(ij\ldots l)} = \text{constant},$$

where $ab\ldots f$ and $ij\ldots l$ are complementary with respect to $2n+1$. We interpret this result as (proportional to) a constant 'energy flux' along the x_{2n+1} axis, with $\mathrm{Im} = 0$ as the condition for no energy loss during the propagation process. It should be noted that the component $e_{(ab\ldots f)}$ and $h_{(ij\ldots l)}$ are complementary components as previously defined. In R^3, we note that

$$-\varepsilon_{a3i}\,\mathrm{Re}(e_a h^*_i) = \mathrm{Re}(e_1 h^*_2 - e_2 h^*_1),$$

proportional to the complex Poynting vector component along the x_3 axis.

It follows that all deductions made when $n = 1$ are applicable for general n, when the fields derived in §§ 6 and 7 are used. These concern relationships between the moduli of reflection and transmission coefficients, results that are given in Heading (1975c, chap 4) when $n=1$, and in more comprehensive forms in Heading (1975b, 1978b). These results are invariant forms and properties when applied to the general kinds of wave propagation discussed in this paper.

9. The anisotropic case

We now introduce a suitable anisotropic medium into equations (16) in place of m. This is achieved by generalising matrix $\boldsymbol{M}$ as used in R^3, as stated in equation (11).

In R^{2n+1}, let n_s denote a constant unit vector with $2n+1$ components. Define the tensor

$$N_{ab\ldots fij\ldots l} = \varepsilon_{ab\ldots fsij\ldots l} n_s,$$

and in keeping with (11), define a general susceptibility tensor M to be

$$M_{ab\ldots fij\ldots l} = \alpha_0 \delta^{ij\ldots l}_{ab\ldots f} + \alpha_1 N_{ab\ldots fij\ldots l} + (\alpha_2/n!) N_{ab\ldots fxy\ldots z} N_{xy\ldots zij\ldots l} + \ldots$$

with as many products as may be necessary, $(n!)^{r-1}$ appearing in the denominator of the term with rN's. The α_r are functions of the one coordinate x_{2n+1} for horizontal stratification.

The generalisation of the particular property given in R^3 when $n=(0,0,1)$, namely, a vertically imposed magnetic field, is achieved by writing $n_s=0$ except $n_{2n+1}=1$, so

$$N_{ab\ldots fij\ldots l} = \varepsilon_{ab\ldots f,2n+1,ij\ldots l}, \tag{28}$$

implying that $ab\ldots f$ and $ij\ldots l$ are complementary suffixes with respect to $2n+1$.

For propagation along the x_{2n+1} axis, (18) is replaced by

$$\begin{aligned} C_{xy\ldots zij\ldots l} h_{ij\ldots l} &= \mathrm{i}\omega\varepsilon_0(1-\alpha_0) e_{xy\ldots z} \\ &\quad - (\mathrm{i}\omega\varepsilon_0/n!)[\alpha_1 N_{xy\ldots zij\ldots l} + (\alpha_2/n!) N_{xy\ldots zab\ldots f} N_{ab\ldots fij\ldots l} + \ldots] e_{ij\ldots l}, \end{aligned}$$

yielding, as for vertical incidence in § 6,

$$\begin{aligned} \mathrm{d}^2_{2n+1} e_{xy\ldots z} &+ k^2 e_{xy\ldots z} - k^2\alpha_0 e_{xy\ldots z} \\ &- k^2[(\alpha_1/n!) N_{xy\ldots zij\ldots l} + (\alpha_2/(n!)^2) N_{xy\ldots zab\ldots f} N_{ab\ldots fij\ldots l} + \ldots] e_{ij\ldots l}. \end{aligned}$$

Now

$$\begin{aligned} N_{xy\ldots zij\ldots l} e_{ij\ldots l} &= \varepsilon_{xy\ldots z,2n+1,ij\ldots l} e_{ij\ldots l} \\ &= n! s e_{(ij\ldots l)}, \end{aligned}$$

where s denotes the sign of $\varepsilon_{xy\ldots z,2n+1,(ij\ldots l)}$, and $ab\ldots f$ and $ij\ldots l$ are complementary with respect to $2n+1$. The brackets may be removed from the suffixes of e, provided we understand that the same sequence of letters is implied in the sign s (with no summation).

Similarly,

$$N_{xy\ldots zab\ldots f} N_{ab\ldots fij\ldots l} e_{ij\ldots l} = \varepsilon_{xy\ldots z,2n+1,ab\ldots f}\, \varepsilon_{ab\ldots f,2n+1,ij\ldots l} e_{ij\ldots l}. \tag{29}$$

Clearly, for given $xy\ldots z$, $ab\ldots f$ must be complementary; moreover, $ij\ldots l$ is complementary to $ab\ldots f$, so must equal $xy\ldots z$ in some order. Hence (29) equals

$$
\begin{aligned}
(n!)^2 & \varepsilon_{xy\ldots z,2n+1,(ab\ldots f)}\varepsilon_{(ab\ldots f),2n+1,(xy\ldots z)}e_{(xy\ldots z)} \\
&= (n!)^2\varepsilon_{xy\ldots z,2n+1,(ab\ldots f)}(-1)^n\varepsilon_{(xy\ldots z),2n+1,(ab\ldots f)}e_{(xy\ldots z)} \\
&= (-1)^n(n!)^2 s^2 e_{xy\ldots z} \\
&= (-1)^n(n!)^2 e_{xy\ldots z}.
\end{aligned}
$$

The differential equation becomes

$$
\begin{aligned}
\mathrm{d}_{2n+1}^2 e_{xy\ldots z} &+ k^2 e_{xy\ldots z} - k^2\alpha_0 e_{xy\ldots z} - k^2\alpha_1 s e_{ij\ldots l} - k^2(-1)^n\alpha_2 e_{xy\ldots z} \\
&- k^2(-1)^n\alpha_3 s e_{ij\ldots l} - k^2\alpha_4 e_{xy\ldots z} - \ldots = 0. \qquad (30)
\end{aligned}
$$

Starting with the α_0 term, the signs are 1, s, $(-1)^n$, $(-1)^n s$, 1, s, $(-1)^n, \ldots$, repeated in groups of four.

Replacing $e_{xy\ldots z}$ by $e_{ij\ldots l}$, and noting that the sign to be introduced is $\varepsilon_{ij\ldots l,2n+1,xy\ldots z} \equiv (-1)^n s$, we have

$$
\begin{aligned}
\mathrm{d}_{2n+1}^2 e_{ij\ldots l} &+ k^2 e_{ij\ldots l} - k^2\alpha_0 e_{ij\ldots l} - k^2(-1)^n s\alpha_1 e_{xy\ldots z} \\
&- k^2(-1)^n\alpha_2 e_{ij\ldots l} - k^2 s\alpha_3 e_{xy\ldots z} - k^2\alpha_4 e_{ij\ldots l} - \ldots = 0, \qquad (31)
\end{aligned}
$$

the successive signs attached to the α's being 1, $(-1)^n s$, $(-1)^n$, s, $1, \ldots$, repeated in groups of four.

Case (i). If $s = \pm 1$ with n even, the terms $e_{xy\ldots z}$ in (30) correspond to $e_{ij\ldots l}$ in (31) in position, and *vice versa*. The combinations

$$u = e_{xy\ldots z} + e_{ij\ldots l}, \qquad v = e_{xy\ldots z} - e_{ij\ldots l}$$

yield non-simultaneous equations.

Case (ii). If $s = \pm 1$ with n odd, the terms $e_{xy\ldots z}$ in (30) correspond to $e_{ij\ldots l}$ in (31), but $e_{ij\ldots l}$ in (30) correspond to $-e_{xy\ldots z}$ in (31). Hence the combinations

$$u = e_{xy\ldots z} + \mathrm{i}e_{ij\ldots l}, \qquad v = e_{xy\ldots z} - \mathrm{i}e_{ij\ldots l}$$

yield non-simultaneous equations.

In case (i), $e_{xy\ldots z} = e_{ij\ldots l}$, $e_{xy\ldots z} = -e_{ij\ldots l}$ for u-waves and v-waves respectively to exist alone. In case (ii), $e_{xy\ldots z} = \mathrm{i}e_{ij\ldots l}$, $e_{xy\ldots z} = -\mathrm{i}e_{ij\ldots l}$ for u-waves and v-waves respectively to exist alone.

This generalises what has been said about equations (12) and (13) when $n = 1$ (odd). The question of polarisation (discussed in relation to equations (13)) appears to be rather artificial, since the field components do not possess specific directions related to the coordinate axes except when $n = 1$. But if pseudo-orthogonal axes $x_{(xy\ldots z)}$, $x_{(ij\ldots l)}$ are defined in a space of dimensions ${}_{2n}C_n$, then with respect to these axes any pair of complementary fields is linearly polarised when n is even, and circularly polarised when n is odd in planes specified by two of these pseudo-axes.

10. Conclusion

We have shown how Maxwell's equations can be generalised so as to refer to a space of $2n+1$ dimensions, with the operations curl and curl curl having appropriate definitions. The second-order wave equations are formulated for the various isotropic modes of propagation, while a particular form of anisotropic propagation is investigated to yield non-simultaneous equations that exhibit the properties of linear and circular polarisation. The main feature of the generalisation is that all the results are very similar to the physical case when $n = 1$, showing the peculiar property of invariance in many unexpected ways.

References

Bremmer H 1949 *Terrestrial Radio Waves* (New York: Elsevier)
Budden K G 1961 *Radio Waves in the Ionosphere* (Cambridge: CUP)
Ginzburg V L 1961 *Propagation of Electromagnetic Waves in Plasma* (Amsterdam: North Holland)
Heading J 1973a *Q. J. Mech. Appl. Math.* **26** 513–28
—— 1973b *Lin. Algb. Applns.* **6** 19–36
—— 1975a *Q. J. Mech. Appl. Math.* **28** 185–206
—— 1975b *Q. J. Mech. Appl. Math.* **28** 427–48
—— 1975c *Ordinary Differential Equations, Theory and Practice* (London: Elek Science)
—— 1975d *Lin. Algb. Applns.* **11** 277–89
—— 1976a *Math. Proc. Camb. Phil. Soc.* **79** 563–71
—— 1976b *Int. J. Math. Edcn. Sci. Tech.* **7** 277–85
—— 1977 *Math. Proc. Camb. Phil. Soc.* **82** 485–8
—— 1978a *J. Inst. Maths. Applns.* **21** 387–400
—— 1978b *Proc. R. Soc.* A **360** 279–300
—— 1979 *Math. Proc. Camb. Phil. Soc.* **85** 361–77
—— 1980 *Wave Motion* **2** 325–34
—— 1981 *J. Phys. A: Math. Gen.* **14** 2881–95
Korn A K and Korn T M (1961) *Mathematical Handbook for Scientists and Engineers* (New York: McGraw-Hill) p 481
Ratcliffe J A 1959 *The Magneto-ionic Theory* (Cambridge: CUP)